Coincidence Problem in Holographic $f(R)$ Gravity

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Abstract

It is well-known that $f(R)$ gravity models formulated in Einstein conformal frame are equivalent to Einstein gravity together with a minimally coupled scalar field. In this case, the scalar field couples with the matter sector and the coupling term is given by the conformal factor. We apply the holographic principle to such interacting models. In a spatially flat universe, we show that the Einstein frame representation of $f(R)$ models leads to a constant ratio of energy densities of dark matter to dark energy.

1 Introduction

It is strongly believed that our universe is in a phase experiencing an accelerated expansion. The simplest candidate to produce this cosmic speed-up is the cosmological constant, the energy density associated with quantum vacuum. However, there are several problems for associating cosmic acceleration with the cosmological constant. First, theoretical estimates on its value are many order of magnitude larger than observations [1]. Second, it is simply a constant, namely that it is not diluted with expansion of the universe. This latter is specifically important in the sense that there are observational evidence [2] demonstrating that the cosmic acceleration is a recent phenomena and the universe must have passed through a deceleration phase in the early stages of its evolution. This deceleration phase is important for successful nucleosynthesis as well as for the structure formation. We therefore need a field evolving during expansion of the universe in such a way that its dynamics makes the deceleration parameter have a signature flip from positive in the early stages of matter dominated era to negative in the present stage [3]. There is also another problem which is the focus of the present note. It concerns with the coincidence between the observed vacuum energy density and the current

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matter density. While these two energy components evolve differently as the universe expands, their contributions to total energy density of the universe in the present epoch are the same order of magnitude.

As a different point of view, cosmic acceleration may be interpreted as evidence either for existence of some exotic matter components or for modification of the gravitational theory. In the first route of interpretation one can take a mysterious cosmic fluid with sufficiently large and negative pressure, dubbed dark energy. These models are usually invoked a scalar field which during its evolution takes negative pressure by rolling down a proper potential. In the second route, however, one attributes the accelerating expansion to a modification of general relativity. A particular class of models that has recently drawn a significant amount of attention is the so-called $f(R)$ gravity models [4]. These models propose a modification of Einstein-Hilbert action so that the scalar curvature is replaced by some arbitrary function $f(R)$.

Recently, different models inspired by holographic principle have been proposed to explain the cosmic acceleration. The basic idea is that the number of degrees of freedom of a physical system scales with its bounding area rather than with its volume [5]. For an effective quantum field theory in a box of size L with an ultraviolet (UV) cutoff Λ, the entropy S scales extensively as $S \sim L^3\Lambda^3$. However, the peculiar thermodynamics of black holes has led Bekenstein [6] to postulate that the maximum entropy in a box of volume L^3 behaves non-extensively, growing as the area of the box. In this sense there is a so-called Bekenstein entropy bound

$$S = L^3\Lambda^3 \leq S_{BH} \equiv \pi L^2 M_p^2 \tag{1}$$

where S_{BH} is the entropy of a black hole of radius L, and $M_p \equiv (8\pi G)^{-\frac{1}{2}}$ stands for the reduced Planck mass. It is important that in this relation the length scale L providing an Infrared (IR) cutoff is determined by the UV cutoff Λ and can not be chosen independently. However, such a non-extensive scaling law seems to provide a breakdown of quantum field theory at large scales. To reconcile this breakdown with the success of local quantum field theory in describing observed particle phenomenology, Cohen et al. [7] proposed a more restrictive bound. Since the maximal energy density in the effective theory is of the order $\rho_\Lambda = \Lambda^4$, requiring that the energy in a given volume not to exceed the energy of a black hole of the same size results in the constraint

$$L^3\rho_\Lambda \leq L M_p^2 \tag{2}$$

If we take the largest value of the length scale L as the IR cutoff saturating the inequality (2), we then obtain the holographic dark energy density

$$\rho_\Lambda = 3c^2 M_p^2 L^{-2} \tag{3}$$

in which $3c^2$ is a numerical constant. It is interesting to note that if the length scale L is characterized by the size of the universe, the Hubble scale H^{-1}, then equation (3) gives a vacuum energy density of the right order of magnitude consistent with observations [7]. It is, however, pointed out that this yields a wrong equation of state parameter for dark energy, and other possible values for L should be chosen such as the size of the future event horizon [8] [9]. This conclusion is, however, based on the assumption that energy densities of dark

energy and dark matter evolve independently. It is shown [10] that, if there is *any* interaction between these two components the identification of L with H^{-1} is possible. In particular, the authors of [10] argued that such an identification necessarily implies a constant ratio of the energy densities of the two components *regardless of the details of the interaction*.
In the present note, we investigate the coincidence problem in the context of holographic $f(R)$ gravity models[†]. In $f(R)$ models the dynamical variable of the vacuum sector is the metric tensor and the corresponding field equations are fourth order. This dynamical variable can be replaced by a new pair which consists of a conformally rescaled metric and a scalar partner. Moreover, in terms of the new set of variables the field equations are those of General Relativity. The original set of variables is commonly called Jordan conformal frame and the transformed set whose dynamics is described by Einstein field equations is called Einstein conformal frame. The dynamical equivalence of Jordan and Einstein conformal frames does not generally imply that they are also physically equivalent. In fact, it is shown that some physical systems can be differently interpreted in different conformal frames [12] [13]. The physical status of the two conformal frames is an open question which we are not going to address here.
We will work in Einstein conformal frame. The motivation is that in this frame there is a coupling between the scalar degree of freedom and matter sector induced by the conformal transformation. In this context, we have already studied the coincidence problem without any use of holographic principle [14]. We have shown that the requirement of a constant ratio of energy densities of the two components, puts some constraints on the functional form of the $f(R)$ function. Here we apply the holographic principle to dark energy density corresponding to the scalar degree of freedom. The IR cutoff is identified with the Hubble scale. We shall show that this interacting holographic dark energy leads to a stationary ratio of energy densities corresponding to dark energy and matter sector in a spatially flat universe regardless of the details of the interaction term. The distinguished feature of the present work is that the interaction term is given by a particular configuration of the $f(R)$ function. We use this fact to argue that Einstein frame representation of *any* holographic $f(R)$ model may address the coincidence problem in a spatially flat universe.

2 The Model

Let us start with introducing the action for an $f(R)$ gravity theory in the Jordan frame

$$S_{JF} = \frac{1}{2}\int d^4x\sqrt{-g}\ M_p^2\ f(R) + S_m(g_{\mu\nu}, \psi) \tag{4}$$

where g is the determinant of $g_{\mu\nu}$ and S_m is the action of (dark) matter which depends on the metric $g_{\mu\nu}$ and some (dark) matter field ψ. Stability in matter sector (the Dolgov-Kawasaki instability [15]) imposes some conditions on the functional form of $f(R)$ models. These conditions require that the first and the second derivatives of $f(R)$ function with respect to the Ricci scalar R should be positive definite. The positivity of the first derivative ensures that the scalar degree of freedom is not tachyonic and positivity of the second derivative tells us that graviton is not a ghost.

[†]Holographic principle has been already applied to different modified gravity models. See, for instance, [11].

It is well-known that $f(R)$ models are equivalent to models in which a scalar field minimally couples to gravity with an appropriate potential function. In fact, we may use a new set of variables

$$\bar{g}_{\mu\nu} = \Omega\ g_{\mu\nu} \tag{5}$$

$$\phi = \frac{M_p}{2\beta} \ln \Omega \tag{6}$$

where $\Omega \equiv \frac{df}{dR} = f^{'}(R)$ and $\beta = \sqrt{\frac{1}{6}}$. This is indeed a conformal transformation which transforms the above action in the Jordan frame to the following action in the Einstein frame [12] [16]

$$S_{EF} = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}\ \{\frac{1}{M_p^2}\bar{R} - \bar{g}^{\mu\nu}\nabla_\mu\phi\ \nabla_\nu\phi - 2V(\phi)\} + S_m(\bar{g}_{\mu\nu}e^{2\beta\phi/M_p}, \psi) \tag{7}$$

All indices are raised and lowered by $\bar{g}_{\mu\nu}$. In the Einstein frame, ϕ is a minimally coupled scalar field with a self-interacting potential which is given by

$$V(\phi(R)) = \frac{M_p^2(Rf'(R) - f(R))}{2f'^2(R)} \tag{8}$$

Note that the conformal transformation induces the coupling of the scalar field ϕ with the matter sector. The strength of this coupling β, is fixed to be $\sqrt{\frac{1}{6}}$ and is the same for all types of matter fields. In the action (7), we take $\bar{g}^{\mu\nu}$ and ϕ as two independent field variables and variations of the action yield the corresponding dynamical field equations. Variation with respect to the metric tensor $\bar{g}^{\mu\nu}$, leads to

$$\bar{G}_{\mu\nu} = M_p^{-2}\ (\bar{T}^{\phi}_{\mu\nu} + \bar{T}^{m}_{\mu\nu}) \tag{9}$$

where

$$\bar{T}^{\phi}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\bar{g}_{\mu\nu}\nabla^\gamma\phi\nabla_\gamma\phi - V(\phi)\bar{g}_{\mu\nu} \tag{10}$$

$$\bar{T}^{m}_{\mu\nu} = \frac{-2}{\sqrt{-\bar{g}}}\frac{\delta S_m(\bar{g}_{\mu\nu}, \psi)}{\delta\bar{g}^{\mu\nu}} \tag{11}$$

are stress-tensors of the scalar field and the matter field system. The trace of (9) is

$$\nabla^\gamma\phi\nabla_\gamma\phi + 4V(\phi) - M_p^2\ \bar{R} = \bar{T}^m \tag{12}$$

which differentially relates the trace of the matter stress-tensor $\bar{T}^m = \bar{g}^{\mu\nu}\bar{T}^m_{\mu\nu}$ to $\bar{R}$. Variation of the action (7) with respect to the scalar field ϕ, gives

$$\Box\phi - \frac{dV(\phi)}{d\phi} = -\frac{\beta}{M_p}\bar{T}^m \tag{13}$$

It is important to note that the two stress-tensors $\bar{T}^m_{\mu\nu}$ and $\bar{T}^{\phi}_{\mu\nu}$ are not separately conserved. Instead, they satisfy the following equations

$$\bar{\nabla}^\mu\bar{T}^m_{\mu\nu} = -\bar{\nabla}^\mu\bar{T}^{\phi}_{\mu\nu} = \frac{\beta}{M_p}\nabla_\nu\phi\ \bar{T}^m \tag{14}$$

We apply the field equations in a spatially flat homogeneous and isotropic cosmology described by Friedmann-Robertson-Walker spacetime

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{15}$$

where $a(t)$ is the scale factor. To do this, we take $\bar{T}^m_{\mu\nu}$ and $\bar{T}^\phi_{\mu\nu}$ as the stress-tensors of a pressureless perfect fluid with energy density $\bar{\rho}_m$, and a perfect fluid with energy density $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$, respectively. In this case, (9) and (13) take the form ‡

$$3H^2 = M_p^{-2}(\rho_\phi + \rho_m) \tag{16}$$

$$2\dot{H} + 3H^2 = -M_p^{-2}\omega_\phi\rho_\phi \tag{17}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\frac{\beta}{M_p}\rho_m \tag{18}$$

where $\omega_\phi = \frac{p_\phi}{\rho_\phi}$ is equation of state parameter of the scalar field ϕ, and overdot indicates differentiation with respect to cosmic time t. The trace equation (12) and the conservation equations (14) give, respectively,

$$\dot{\phi}^2 + M_p^2\ R - 4V(\phi) = \rho_m \tag{19}$$

$$\dot{\rho}_m + 3H\rho_m = Q \tag{20}$$

$$\dot{\rho}_\phi + 3H(\omega_\phi + 1)\rho_\phi = -Q \tag{21}$$

where

$$Q = \frac{\beta}{M_p}\dot{\phi}\rho_m \tag{22}$$

is the interaction term. This term vanishes only for $\phi =$ const., which due to (6) it happens when $f(R)$ linearly depends on R. The direction of energy transfer depends on the sign of Q or $\dot{\phi}$. For $\dot{\phi} > 0$, the energy transfer is from dark energy to dark matter and for $\dot{\phi} < 0$ the reverse is true§.

Let us consider time evolution of the ratio $r \equiv \rho_m/\rho_\phi$,

$$\dot{r} = \frac{\dot{\rho}_m}{\rho_\phi} - r\frac{\dot{\rho}_\phi}{\rho_\phi} \tag{23}$$

If we combine the latter with the balance equations (20) and (21), we obtain

$$\dot{r} = 3Hr[\omega_\phi + (1 + \frac{1}{r})\frac{\Gamma}{3H}] \tag{24}$$

‡Hereafter we will use unbarred characters in the Einstein frame.

§Dark energy and dark matter are the most important energy/mass components contained in the universe. However, there is no experiment to show that these components interact with ordinary matter systems. It is quite possible that these components interact with each other while not being coupled to standard model particles.

where

$$\Gamma = \frac{Q}{\rho_\phi} = \frac{\beta}{M_p} r\dot{\phi} \tag{25}$$

is the decay rate. Now we apply the holographic relation to dark energy density ρ_ϕ with $L = H^{-1}$,

$$\rho_\phi = 3c^2 M_p^2 H^2 \tag{26}$$

where c^2 is a numerical constant introduced for convenience. This gives

$$\dot{\rho}_\phi = 6c^2 M_p^2 H\dot{H} \tag{27}$$

We combine (26) with (17) to obtain

$$\dot{H} = -\frac{3}{2}H^2(1 + \frac{\omega_\phi}{r+1})(r+1)c^2 \tag{28}$$

One can easily check that

$$c^2 = \frac{1}{r+1} \tag{29}$$

which reduces (28) to

$$\dot{H} = -\frac{3}{2}H^2(1 + \frac{\omega_\phi}{r+1}) \tag{30}$$

Substituting this into (27) gives

$$\dot{\rho}_\phi = -9c^2 M_p^2 H^3(1 + \frac{\omega_\phi}{r+1}) \tag{31}$$

When we put the latter together with the holographic relation (26) into the balance equation (21), we obtain

$$\omega_\phi = -(1 + \frac{1}{r})\frac{\Gamma}{3H} \tag{32}$$

This yields the equation of state parameter in terms of r and the decay rate Γ. Note that there is no non-interacting limit in our case since $\Gamma = 0$ corresponds to $\phi =$ const., or equivalently, ΛCDM model.

From the expression (32), it is clear that when $\frac{\Gamma}{3H} << 1$ the equation of state of dark energy is closely related to that of the dust. This can also be seen from the balance equation (21). In the other limiting case, when $\frac{\Gamma}{3H} >> 1$ one takes $\omega_\phi << -1$. This behavior correspond to a signature flip of the deceleration parameter. To see this, we write the deceleration parameter as,

$$q = -1 - \frac{\dot{H}}{H^2} \tag{33}$$

This relation together with (30) and (32) results in

$$q = \frac{1}{2}(1 - \frac{\Gamma}{rH}) \tag{34}$$

which in the above two limiting cases changes the sign from $q > 0$ to $q < 0$, respectively.

As our main observation, we remark that if one uses (32) in the relation (24) one then takes

$\dot{r} = 0$ or $r =$ constant. The reasoning is simple : from the holographic relation (26) one infers that ρ_ϕ scales like the critical density $\rho_c = 3M_p^2 H^2$. As a consequence, the density parameter corresponding to ϕ must be a constant so that $\Omega_\phi = \frac{\rho_\phi}{\rho_c} = c^2$. With this result, the Friedmann equation $\Omega_\phi + \Omega_m = 1$ results in $\Omega_m = 1 - c^2$. Thus ρ_m has the same scaling as ρ_ϕ and the ratio r is a constant. There are some remarks to do with respect to this result. First, it is independent of the details of the decay rate Γ or the interaction Q. Since the interaction is given by the shape of the $f(R)$ function we conclude that applying holographic principle to the dark energy density $\rho_\phi = 3c^2 M_p^2 H^2$ necessarily leads to a constant ratio of energy densities $r = \rho_m/\rho_\phi$, irrespective of the form of the $f(R)$ function¶. Second, it is the consequence of the identification $L = H^{-1}$. If one takes other length scales such as event horizon $L_e = a(t) \int_t^\infty \frac{dt'}{a(t')}$ or particle horizon $L_p = a(t) \int_0^t \frac{dt'}{a(t')}$ as the IR cutoff, then scaling of ρ_ϕ will be different from that of the critical density and the ratio r is no longer stationary. In contrary to the ratio r, accelerating expansion requires particular configurations of $f(R)$ functions. This is clear from the expression (34) which the requirement that $q < 0$ automatically sets a constraint on the decay rate.

It is quite possible that the constancy of the ratio of energy densities r is a recent phenomenon. The two energy components can evolve differently as the universe expands until the present epoch which their contributions to total energy density takes a constant configuration and of the same order of magnitude. To model this behavior, we assume that during evolution of the universe the holographic relation holds as the unsaturated form $\rho_\phi \leq M_p^2 H^2$ and the saturated relation (26) is a recent phenomenon. This is equivalent to assume that

$$\rho_\phi = 3\alpha(t) M_p^2 H^2 \tag{35}$$

where $\alpha(t) \leq c^2$ with $\alpha(t)$ being a parameter which evolves with cosmic expansion. Note that the IR cutoff does not change and remains $L = H^{-1}$. In fact, variation of $\alpha(t)$ in the relation (35) characterizes the degree of the saturation in the holographic bound $\rho_\phi \leq M_p^2 H^2$.

In this case, the relation (32) takes the form

$$\omega_\phi = -(1 + \frac{1}{r})(\frac{\Gamma}{3H} + \frac{\dot{\alpha}}{3H\alpha}) \tag{36}$$

We may combine the latter with (24) to obtain

$$\frac{\dot{\alpha}}{\alpha} = -\frac{\dot{r}}{r+1} \tag{37}$$

This has a solution $\alpha(t) = \frac{1}{r+1}$ which is compatible with (29) up to an integration constant. Since $\dot{\alpha} > 0$ by construction, r should be a decreasing function of time. This behavior allows ω_ϕ in (36) to be more negative compared with the case that α=constant.

¶There are different constraints on the configuration of a viable $f(R)$ function, such as constraints coming from Dolgov-Kawasaki instability issue [15] or constraints related to local gravity experiments [17]. However, resolution of the coincidence problem in our analysis does not put any constraint on the form of the $f(R)$ function.

3 Conclusion

We have considered the Einstein frame representation of a general $f(R)$ gravity model and apply the holographic relation to the energy density ρ_ϕ corresponding to the scalar degree of freedom of the metric tensor. Taking the Hubble radius as the IR cutoff, we observe that the ratio $r = \rho_m/\rho_\phi$ takes a constant configuration for any $f(R)$ function in a spatially flat universe. It should be noted that the choice $L = H^{-1}$ attributes an energy density to ρ_ϕ consistent with observations [7]. Thus the two different features of the cosmological constant problem, namely the fine tuning and the cosmic coincidence problems, may be addressed in this context.

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